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An invariant operator due to F Klein quantizes H Poincaré's dodecahedral 3-manifold

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Abstract

The eigenmodes of the Poincaré dodecahedral 3-manifold M are constructed as eigenstates of a novel invariant operator. The topology of M is characterized by the homotopy group $\pi_1(M)$, given by loop composition on M, and by the isomorphic group of deck transformations $\mathrm{deck}(\tilde{M})$, acting on the universal cover $\tilde{M}$. $(\pi_1(M), \tilde{M})$ are known to be the binary icosahedral group $\mathcal{H}_3$ and the sphere S^3, respectively. Taking S^3 as the group manifold $SU(2, C)$ it is shown that $\mathrm{deck}(\tilde{M}) \sim \mathcal{H}_3^r$ acts on $SU(2, C)$ by right multiplication. A semidirect product group is constructed from $\mathcal{H}_3^r$ as a normal subgroup and from a second group $\mathcal{H}_3^c$ which provides the icosahedral symmetries of M. Based on F Klein's fundamental icosahedral $\mathcal{H}_3$-invariant, we construct a novel Hermitian $\mathcal{H}_3$-invariant polynomial (generalized Casimir) operator $\mathcal{K}$. Its eigenstates with eigenvalues κ quantize a complete orthogonal basis on Poincaré's dodecahedral 3-manifold. The eigenstates of lowest degree $\lambda = 12$ are 12 partners of Klein's invariant polynomial. The analysis has applications in cosmic topology [15, 18]. If the Poincaré 3-manifold M is assumed to model the space part of a cosmos, the observed temperature fluctuations of the cosmic microwave background must admit an expansion in eigenstates of $\mathcal{K}$.

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1. Introduction

We present a rigorous Lie algebraic operator approach to the eigenmodes of the Poincaré dodecahedral 3-manifold. The eigenmodes are replaced by eigenstates of an operator $\mathcal{K}$ invariant under deck transformations. They are quantized by their eigenvalues. These eigenstates on M form a complete orthogonal basis appropriate for the expansion of observables.

(I) Given a topological manifold X, the fundamental or first homotopy group $\pi_1(X)$ is formed by loops and their composition. There is a manifold $\tilde{X}$, called the universal cover of X, which is simply connected and on which a group $\mathrm{deck}(\tilde{X})$ isomorphic to $\pi_1(X)$ acts discontinuously by so-called deck transformations. The quotient set $\tilde{X}/\mathrm{deck}(\tilde{X})$ is the original X. For $X = M$ being the Poincaré dodecahedral 3-manifold, the universal cover is the 3-sphere $\tilde{M} = S^3$, and the fundamental group is $\pi_1(M) = \mathcal{H}_3$, the binary icosahedral group. We use the notation $\mathcal{H}_3$ to avoid confusion with the symbol H_3 [8], p 46, for the icosahedral Coxeter group.

(II) The functional analysis on these topological manifolds starts on the universal cover and employs representation theory. As eigenmodes of S^3 we take the spherical harmonics, homogeneous solutions of fixed degree λ of the Laplace equation. They transform according to particular irreducible representations (irreps) $D^j \times D^j$, $j = \lambda/2$ of the full group of isometries $SO(4, R)$ of S^3. The deck transformations form a subgroup $\mathcal{H}_3 < SO(4, R)$ of isometries. We employ the subduction of irreps under the restriction $SO(4, R) > \mathcal{H}_3$ to decompose the spherical harmonics on S^3 into modes transforming under the irreps D^α of $\mathcal{H}_3$. This subduction is solved with a Lie-algebraic novel operator technique. Among these modes we determine the subset transforming according to the identity irrep D^{α_0} of $\mathcal{H}_3$. Taken as functions on S^3 they are $\mathcal{H}_3$-periodic with respect to the decomposition of S^3 into copies of M. Their unique restrictions from S^3 to M are the eigenmodes of M.

The analysis is carried out as follows: in section 2, we set up the continuous geometry of the universal cover S^3. We take S^3 in the equivalent form of the group manifold $SU(2, C)$ and implement the action of $SO(4, R)$. In section 3, we introduce discrete subgroups of $SO(4, R)$ which provide all the discrete group actions needed. In section 4, we show that the group of deck transformations $\mathrm{deck}(\tilde{M}) \sim \mathcal{H}_3$ is the normal subgroup in a semidirect product, lemma 2. The elements of $\mathcal{H}_3$ correspond to Hamilton's icosians. We show that $\mathrm{deck}(\tilde{M}) = \mathcal{H}_3$ acts on $SU(2, C)$ by right action, lemma 3, and interprete the semidirect product in terms of symmetries of M, lemma 4. The spherical harmonics on S^3 are identified in section 5 as Wigner's D-functions on $SU(2, C)$, lemma 5. From Felix Klein's fundamental icosahedral invariant [10] in section 6 we obtain 12 more $\mathcal{H}_3$-invariant polynomials of degree 12. They are classified in terms of left action generators in lemma 7. One of these invariant polynomials due to Klein allows to pass in sections 7, 8 to a invariant Hermitian generalized Casimir operator $\mathcal{K}$ on the enveloping Lie algebra of $SO(4, R)$, lemma 8. It characterizes the irrep subduction $SO(4, R) > \mathcal{H}_3$. The operator is shown to quantize the $\mathcal{H}_3$-invariant eigenmodes of the Poincaré 3-manifold M as eigenstates, theorem 1. The term quantization is used here in the sense given in Schrödinger's papers [25] on quantization as an eigenvalue problem.

We analyse the spectrum of $\mathcal{K}$ and completely resolve its degeneracy by additional Hermitian operators and quantum numbers. The eigenstates of $\mathcal{K}$ up to $j = 6$ are explicitly given in algebraic form. Analysis of the spectrum of $\mathcal{K}$ yields the selection rules for eigenmodes on M versus those on $\tilde{M} = S^3$. A comparison to related work from cosmic topology is given in section 11. Operator symmetrizations appearing in $\mathcal{K}$ are carried out in the appendix.

2. Continuous geometry of S^3

The sphere S^3, embedded in Euclidean space E^4 as a manifold, is in one-to-one correspondence to the group manifold $SU(2, C)$. Let $(x_0, x_1, x_2, x_3) \in E^4$ be orthogonal coordinates and define

$$u(x) := \begin{bmatrix} x_0 - \mathrm{i}x_3 & (-\mathrm{i})(x_1 - \mathrm{i}x_2) \\ (-\mathrm{i})(x_1 + \mathrm{i}x_2) & (x_0 + \mathrm{i}x_3) \end{bmatrix} = x_0\sigma_0 + (-\mathrm{i})\sum_{j=1}^{3} x_j\sigma_j,$$
$$\det(u(x)) = \sum_{0}^{3}(x_i)^2 = 1. \tag{1}$$

We shall employ complex variables and rewrite equation (1) as

$$u(x) = u(z, \overline{z}) = \begin{bmatrix} z_1 & z_2 \\ -\overline{z}_2 & \overline{z}_1, \end{bmatrix}, \qquad z_1\overline{z}_1 + z_2\overline{z}_2 = 1. \tag{2}$$

In equation (1), σ_0 denotes the 2×2 unit matrix and σ_j the Pauli matrices. The four Cartesian coordinates in equation (1) could be replaced by three independent real Euler (half)-angles $(\alpha, \beta, \gamma), 0 \leqslant (\alpha, \gamma) < 4\pi, 0 \leqslant \beta < 2\pi$, see [4], pp 53–67. To fully cover S^3, the Euler angular parameters must take their full range of values. Equation (1) may be considered as the physicist's version of an expression employing quaternions, given in a different context in [20, 22] as

$$\begin{aligned} &\mathbf{x} = \mathbf{1}x_1' + \mathbf{i}x_2' + \mathbf{j}x_3' + \mathbf{k}x_4', \\ &\mathbf{1} = \sigma_0, \qquad \mathbf{i} = \mathrm{i}\sigma_3, \qquad \mathbf{j} = \mathrm{i}\sigma_2, \qquad \mathbf{k} = \mathrm{i}\sigma_1, \\ &(x_1', x_2', x_3', x_4') \to (x_0, -x_3, -x_2, -x_1). \end{aligned} \tag{3}$$

The second and third line convert the scheme of equation (3), first line, to the one used in equation (1). Any matrix $u(x) \in SU(2, C)$ is mapped one-to-one to a point on S^3 by equations (1), (2).

For two matrices $u(y), u(x)$ of the type equation (1) we define the Hermitian scalar product by

$$\langle u(y), u(x)\rangle := \tfrac{1}{2}\,\mathrm{Trace}(u(y)^\dagger \times u(x)) = y_0x_0 + \sum_{j=1}^{3} y_jx_j. \tag{4}$$

Consider group actions of the type $SU(2, C) \times SU(2, C) \to SU(2, C)$. The left, right and conjugation actions of $g_l, g_r, g_c \in SU^{l,r,c}(2, C)$ on $u \in SU(2, C)$ are defined by

$$(g_l, u) \to g_l^{-1}u, \qquad (g_r, u) \to ug_r, \qquad (g_c, u) \to g_c^{-1}ug_c. \tag{5}$$

The left and right actions by $SU^{l,r}(2, C)$ commute. When combined they yield a direct product action $SU^l(2, C) \times SU^r(2, C)$ linear in the matrix elements of u which is easily shown to fully cover $SO(4, R)$. Removing the elements $(I, -I) := (\sigma_0, -\sigma_0)$ of the stability group Z_2 from the action of $SU^l(2, C) \times SU^r(2, C)$ on $SU(2, C)$, we have the well-known result

$$SO(4, R) = (SU^l(2, C) \times SU^r(2, C))/Z_2. \tag{6}$$

We turn to the conjugation action. By writing u according to equation (1), one sees that x_0 is unchanged while the conjugation of the Pauli matrices yields

$$g_c^{-1}\sigma_j g_c = \sum_{i=1}^{3} \sigma_i D^1_{ij}(g_c), \tag{7}$$

so that $(x_0, x_1, x_2, x_3) \in E^4$ transform according to $(1 \times D^1)$, with D^1 the defining real irrep of the rotation group $SO(3, R)$.

Combining right multiplication and conjugation, define a multiplication rule

$$(g_{r_1}, g_{c_1}) \times (g_{r_2}, g_{c_2}) := (g_{c_2}^{-1}g_{r_1}g_{c_2}g_{r_2}, g_{c_1}g_{c_2}). \tag{8}$$

This multiplication rule is associative and generates a group. Subgroups $SU^r(2, C)$, $SU^c(2, C)$ are generated by the elements $(g_r, e), (e, g_c)$, respectively. One finds the conjugation rule

$$(e, g_c)(g_r, e)(e, g_c^{-1}) = (g_cg_rg_c^{-1}, e), \tag{9}$$

which shows that equation (8) yields a semidirect product

$$SU^r(2, C) \times_s SU^c(2, C), \tag{10}$$

with $SU^r(2, C)$ the normal subgroup. We let this semidirect product act on $u \in SU(2, C)$ as

$$((g_r, g_c), u) \rightarrow g_c^{-1} u g_c g_r. \tag{11}$$

This action, with emphasis on the right action, was the reason for the choice, equation (8), and will be used in what follows. The action is linear in the matrix elements of u and from equation (2) yields a homomorphism from the semidirect product group (equation (10)) to $SO(4, R)$. Since $(g_r, \pm g_c)$ yield the same element of $SO(4, R)$, the homomorphism is two-to-one, and from the stability group under the action (equation (11)) we find

$$SO(4, R) = (SU^r(2, C) \times_s SU^c(2, C))/Z_2. \tag{12}$$

From the group actions, we infer the relevant representation theory: the correspondence of $SO(4, R)$ to the direct product, equation (6), implies that its general irreducible representations (irreps) are the direct products $D^{j_1} \times D^{j_2}$ of two independent irreps of $SU^{l,c}(2, C)$. The sphere S^3 can be obtained by acting with $SO(4, R)$ on the representative point $(1, 0, 0, 0)$. This point has the stability group $SO(3, R)$, acting according to equation (7). The stability group corresponds to the second factor (conjugation action) of the semidirect product action, equation (11). It then follows that the homogeneous or coset space has the structure

$$S^3 := SO(4, R)/SO(3, R) = SU(2, C), \tag{13}$$

in line with equation (1) and corresponding to the first factor (right action) in the semidirect product, equation (10).

In relation with the Laplace equation on E^4, we are interested in the spherical harmonics on S^3 and their transformation under $SO(4, R)$. They will be identified as the Wigner representation functions $D^j(u)$, $u \in SU(2, C)$ in section 5 and shown to transform according to the irreps $D^j \times D^j$ of $SU^l(2, C) \times SU^r(2, C)$.

3. Discrete and Coxeter groups acting on S^3

Coxeter groups [8], chapter 5, are finitely generated by reflections such that any product of two reflection generators is of finite order. We shall show in the next sections that the discrete group of deck transformations and the symmetry group of the dodecahedral Poincaré 3-manifold appear as subgroups of a spherical Coxeter group.

Consider the spherical Coxeter group [8] with four generators R_1, R_2, R_3, R_4, Coxeter–Dynkin diagram and relations

$$\circ \overset{5}{-} \circ \overset{3}{-} \circ \overset{3}{-} \circ$$

$$\begin{aligned} &:= \langle R_1, R_2, R_3, R_4 | \\ &(R_1)^2 = (R_2)^2 = (R_3)^2 = (R_4)^2 = (R_1 R_2)^5 = (R_2 R_3)^3 = (R_3 R_4)^3 = e\rangle. \end{aligned} \tag{14}$$

This Coxeter group admits an isometric action on E^4 by Weyl reflections

$$W(a_i)x := x - 2\frac{\langle x, a_i\rangle}{\langle a_i, a_i\rangle} a_i \tag{15}$$

in four hyperplanes perpendicular to four Weyl vectors $a_1, \ldots, a_4$. The action maps S^3 to S^3. We choose these vectors as

$$\begin{aligned}
a_1 &= (0, 0, 1, 0),\\
a_2 &= \left(0, -\frac{\sqrt{-\tau+3}}{2}, \frac{\tau}{2}, 0\right),\\
a_3 &= \left(0, -\sqrt{\frac{\tau+2}{5}}, 0, -\sqrt{\frac{-\tau+3}{5}}\right),\\
a_4 &= \left(\frac{\sqrt{2-\tau}}{2}, 0, 0, -\frac{\sqrt{\tau+2}}{2}\right),\\
\langle a_i, a_i\rangle &= 1, \qquad i = 0, \ldots, 3, \qquad \langle a_1, a_2\rangle = \cos(\pi/5) = \tau/2,\\
\langle a_2, a_3\rangle &= \cos(\pi/3) = 1/2, \qquad \langle a_3, a_4\rangle = \cos(\pi/3) = 1/2,\\
\langle a_i, a_j\rangle &= 0 \quad \text{otherwise}, \qquad \tau = (1+\sqrt{5})/2.
\end{aligned} \tag{16}$$

Any Weyl reflection W in the Coxeter group, equation (14), is represented by a 4×4 real matrix of determinant $\det(W) = -1$ and so does not belong to $SO(4, R)$. We shall work with the normal subgroup of the Coxeter group given by

$$S(\circ\overset{5}{-}\circ\overset{3}{-}\circ\overset{3}{-}\circ) := (\circ\overset{5}{-}\circ\overset{3}{-}\circ\overset{3}{-}\circ) \cap SO(4, R). \tag{17}$$

All elements of this normal subgroup contain an even number of reflections when written as products of generators. They can be generated from the products $(R_i R_{i+1})$, $i = 1, 2, 3$.

The subgroup

$$\circ\overset{5}{-}\circ\overset{3}{-}\circ = \langle R_1, R_2, R_3 | (R_1)^2 = (R_2)^2 = (R_3)^2 = (R_1R_2)^5 = (R_2R_3)^3 = e\rangle \tag{18}$$

is the icosahedral Coxeter subgroup H_3 including reflections and of order 120. Any element g of this group leaves x_0 unchanged and acts on E^4 as $(1 \times D^1(g))$.

The even normal subgroup of the icosahedral Coxeter group, equation (18), we denote by

$$S(\circ\overset{5}{-}\circ\overset{3}{-}\circ) := (\circ\overset{5}{-}\circ\overset{3}{-}\circ) \cap SO(4, R), \tag{19}$$

it is of order 60 and consists of all icosahedral rotations. In particular the even element (R_1R_2) stabilizes the vector $(0, 0, 0, 1) \in E^4$. Given any element g_c of the binary icosahedral group $\mathcal{H}_3^c$, its conjugation map, equation (5), on E^4 yields a two-to-one homomorphism $(1, g_c) \to (1 \times D^1(g_c)) \in S(\circ\overset{5}{-}\circ\overset{3}{-}\circ) < SO(4, R)$ which fully covers the icosahedral group, equation (19).

Restricting the two groups in the continuous semidirect product (equation (10)) with multiplication rule (equation (8)) to two binary icosahedral groups, we define a semidirect product

$$\mathcal{H}_3^r \times_s \mathcal{H}_3^c, \tag{20}$$

with normal subgroup $\mathcal{H}_3^r$ and homomorphic to an $SO(4, R)$ action (equation (11)) on S^3.

Lemma 1 (Discrete groups acting on S^3)**.** *The semidirect product group (equation (20)) admits a two-to-one homomorphism to* $S(\circ\overset{5}{-}\circ\overset{3}{-}\circ\overset{3}{-}\circ)$*, equation 17.*

Proof. The group $S(\circ\overset{5}{-}\circ\overset{3}{-}\circ\overset{3}{-}\circ)$ may be generated by the even products (R_1R_2, R_2R_3, R_3R_4). Preimages in $\mathcal{H}_3^r \times_s \mathcal{H}_3^c$ of the first two products can be constructed from elements $(e, g_c) \in \mathcal{H}_3^c$.

For (R_3R_4) it suffices to construct the preimage of some even element (QR_4), $Q \in \circ\overset{5}{-}\circ\overset{3}{-}\circ$, since then $(R_3R_4) = (R_3Q^{-1})(QR_4)$, $(R_3Q^{-1}) \in S(\circ\overset{5}{-}\circ\overset{3}{-}\circ)$. □

Such a preimage from $\mathcal{H}_3^r$ will be constructed in lemma 2.

4. The homotopy group and the group of deck transformations of the Poincaré 3-manifold $\tilde{M}$

For general topological notions we refer to [21, 24, 26]. The topology of a manifold is characterized by its homotopy and homology groups. It is well known, [26] p 217, that the first homology groups of M and S^3 are trivial and therefore fail to discriminate the topologies of these two manifolds. The homotopy group $\pi_1(M)$ acts by loop composition. It was constructed and analysed by Seifert and Threlfall [27], pp 216–8, using loops along the edges of the dodecahedron M, and shown by Threlfall in [28] from its generators and relations to be isomorphic to the binary icosahedral group $\mathcal{H}_3$ of order $|\mathcal{H}_3| = 120$. The order comes about by the two-to-one mapping between the binary and the proper three-dimensional icosahedral group of order 60. The group $\mathcal{H}_3$ is interpreted geometrically in [28].

To begin with we construct on S^3 a spherical dodecahedron M. Consider the Weyl vector a_4 from equation (16). The Weyl hyperplane $\langle x, a_4\rangle = 0$ in E^4, invariant under the Weyl reflection $W(a_4)$, intersects S^3 in a unit sphere S^2. The vector a_4 and its Weyl hyperplane under the icosahedral rotation group $S(\circ\overset{5}{-}\circ\overset{3}{-}\circ)$ each have 12 images in E^4 and 12 corresponding unit spheres S^2 on S^3. These 12 unit spheres bound a convex spherical dodecahedron M on S^3. We label the 12 faces of the spherical dodecahedron M as follows: the spherical face in the hyperplane perpendicular to a_4 we denote as $\partial_1 M$, the five faces sharing an edge with $\partial_1 M$ we label counterclockwise by $\partial_2 M, \ldots, \partial_6 M$, and the faces opposite to $\partial_1 M, \ldots, \partial_6 M$ by $\partial_{\bar{1}} M, \ldots, \partial_{\bar{6}} M$. The spherical dodecahedron differs from the Euclidean dodecahedron, compare [24], p 35: the faces are spherical, faces adjacent to an edge have dihedral angle $2\pi/3$, any edge becomes a three-fold rotation axis generated by (R_3R_4) or its conjugates. In the next equation, we relate the enumeration of faces [11, 12] to the letters used in [26, 27]:

$$\begin{array}{lcccccc} [26, 27]: & A & B & C & D & E & F. \\ [11]: & 1 & 5 & 6 & 2 & 3 & 4. \end{array} \tag{21}$$

Since the icosahedral Coxeter group (equation (18)) permutes the faces of M, the generators of $\circ\overset{5}{-}\circ\overset{3}{-}\circ$ can be written as signed permutations in cycle notation as follows [11, 12]:

$$R_1 = (23)(46), \qquad R_2 = (24)(56), \qquad R_3 = (15)(2\bar{3}). \tag{22}$$

We multiply permutations from right to left. The icosahedral rotations of the group $S(\circ\overset{5}{-}\circ\overset{3}{-}\circ)$ (equation (19)) provide the symmetry of the dodecahedron M.

The classical prescription for constructing from the spherical dodecahedron the Poincaré dodecahedral 3-manifold M, given by Weber, Seifert and Threlfall in [26, 27], is the gluing of opposite faces of the dodecahedron after a rotation by an angle $\pi/5$.

The group of deck transformations $\mathrm{deck}(\tilde{X})$ [26], pp 196–7, [21], p 398, is defined as the group G acting on the universal cover $\tilde{X}$ such that $\tilde{X}/\mathrm{deck}(\tilde{X})$ equals X. By general theorems given in [26], pp 181–98, the groups $\mathrm{deck}(\tilde{X})$ and $\pi_1(X)$ are isomorphic. In the present case the action of $\mathrm{deck}(\tilde{M}) \sim \mathcal{H}_3$ tiles the universal cover $\tilde{M} = S_3$ by dodecahedral copies of M, produces the face-to-face gluing conditions, and forms the 120-cell [23], pp 172–6.

We construct $\mathrm{deck}(\tilde{M})$ within the geometry of S^3 given in section 2 by the use of the group equations (17), (20). By shifting, rotating and gluing a copy C_1M to the face $\partial_1 M$ of M we find within the groups (equations (17), (20)), a first generator $C_1 \in \mathrm{deck}(\tilde{M})$.

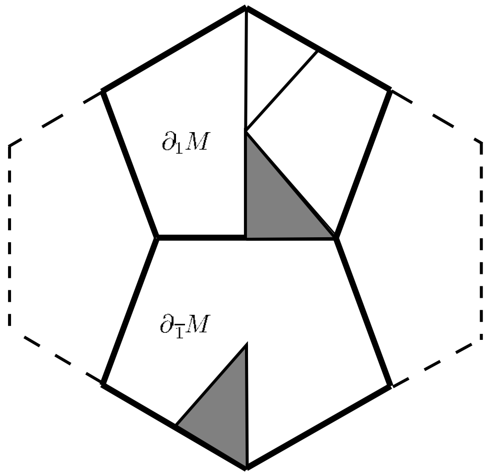

Figure 1. Face-to-face gluing for the dodecahedral Poincaré manifold M as a product of three operations from the Coxeter group (equation (14)): two opposite pentagonal faces of the dodecahedron M (dashed lines) are shown in a projection along a two-fold axis perpendicular to the figure. (1) The shaded triangle of face $\partial_{\bar{1}} M$, bottom pentagon, is mapped by the inversion P to the white triangle on the opposite face $\partial_1 M$, top pentagon. (2) The white triangle is mapped by a counterclockwise rotation $5_{\bar{1}}^{-2}$ by $-4\pi/5$ about the axis passing through the midpoints of both faces to its shaded final position. (3) The Weyl generator R_4 reflects M in the hypersurface containing face $\partial_1 M$ and maps M to its face-to-face neighbour $R_4 M$. The product $C_1 = R_4 5_{\bar{1}}^{-2} P$ reproduces the counterclockwise rotation by $\pi/5$, shift between opposite faces and gluing prescribed in [26, 27] as a generator of the homotopy group.

Denote the five-fold rotation around the midpoint of face $\partial_1 M$ by $5_1 = (R_2 R_1) = (23456)$ in cycle notation (equation (22)). Consider the following product of elements from the icosahedral Coxeter group: the inversion $P := (1\bar{1})(2\bar{2})(3\bar{3})(4\bar{4})(5\bar{5})(6\bar{6})$, followed by a rotation $(5_1)^{-2} := (53642)$, followed by R_4. These operations are illustrated in figure 1. Note that P commutes with all elements of the icosahedral Coxeter group. We claim

Lemma 2 (The group of deck transformations is the normal subgroup $\mathcal{H}_3^r$ of the semidirect product group (equation (20)))**.** *The element*

$$C_1 := R_4 5_{\bar{1}}^{-2} P \tag{23}$$

in the group (equation (17)) is the generator C_1 of $\mathrm{deck}(\tilde{M})$ *correponding to shifting, rotating M by $\pi/5$ and gluing face $\partial_{\bar{1}}(C_1 M)$ to face $\partial_1 M$ in S^3.*

Proof. The element C_1 of equation (23) contains two reflections R_4 and P of determinant $\det = -1$ and so belongs to the normal subgroup (equation (17)) of the full Coxeter group. In geometric terms it is clear that the operation C_1 of equation (23) reproduces, in terms of elements of the group (equation (17), the prescription for rotating and gluing given in [26, 27], see figure 1.

We now analyse the correspondence of C_1 (equation (23)) to an element of the group of deck transformations $\mathrm{deck}(\tilde{M}) \sim \mathcal{H}_3$. For this purpose we determine the action of $g = \left(R_4, 5_{\bar{1}}^{-2}, P, C_1\right)$ in the geometry of section 2. In terms of mappings of the complex parameters (equation (2)) and their composition one finds for the elements

$(g,(z_1,z_2)) \to \mathrm{im}_g(z_1,z_2) = (z_1',z_2')$

$$
\begin{array}{ll}
g: & (z_1',z_2') = \mathrm{im}_g(z_1,z_2), \\
R_4: & (-\epsilon^2\overline{z}_1, z_2), \\
5_1^{-2}: & (z_1, \epsilon^2 z_2), \\
P: & (\overline{z_1}, -z_2), \\
R_4 5_1^{-2} P: & (-\epsilon^{-2} z_1, -\epsilon^2 z_2), \\
\epsilon := \exp(2\pi \mathrm{i}/5). &
\end{array}
\qquad , \tag{24}
$$

The action of C_1 (equation (23)) on (z_1,z_2) from equation (24) can be rewritten in terms of a right action on $u(x)$,

$$
\begin{aligned}
&C_1 : u(x) \to u(x)v, \\
&v = \begin{bmatrix} -\epsilon^{-2} & 0 \\ 0 & -\epsilon^2 \end{bmatrix} \in \mathcal{H}_3^r.
\end{aligned} \tag{25}
$$

That $v \in \mathcal{H}_3^r$ follows by comparison with Klein's list of elements of the binary icosahedral group given in equation (26). □

Any other element of $\mathrm{deck}(\tilde{M})$, corresponding to the gluing of another pair of faces, can be obtained within the semidirect product group (equation (20)) by conjugation according to equation (9) of $v \in \mathcal{H}_3^r$ (equation (25)) with an element $g_c \in \mathcal{H}_3^c$, homomorphic to an element of the icosahedral group $S(\circ\frac{5}{}\circ\frac{3}{}\circ)$. In particular one finds $(C_i)^{-1} = C_{\bar{i}}$. It is a nontrivial result of [27, 28] that the special elements corresponding to 6 gluings generate the binary icosahedral group $\mathcal{H}_3^r$ isomorphic to $\mathrm{deck}(\tilde{M})$ acting on S^3. Therefore we find

Lemma 3 (The group of deck transformations acts from the right on $S^3 \sim SU(2,C)$)**.** *The group of deck transformations* $\mathrm{deck}(\tilde{M}) \sim \mathcal{H}_3$ *acts on* S^3 *in the parametrization (equation (1)) by right multiplication* $u(x) \to ug_r, g_r \in \mathcal{H}_3^r$ *with the elements* g_r *given by Klein as in equation (26).*

Lemma 4 (Symmetry and group of deck transformations for M)**.** *The semidirect product group* $\mathcal{H}_3^r \times_s \mathcal{H}_3^c$ *(equation (10)) is associated with the Poincaré dodecahedral 3-manifold M. Under the two-to-one homomorphism* $\mathcal{H}_3^r \times_s \mathcal{H}_3^c \to S(\circ\frac{5}{}\circ\frac{3}{}\circ\frac{3}{}\circ) < SO(4,R)$, *the groups* $\mathcal{H}_3^r$ *and* $\mathcal{H}_3^c$ *provide the groups of deck transformations and of icosahedral symmetries, respectively, on* $\tilde{M}$.

Klein [10], pp 41–2, gives the elements of $\mathcal{H}_3$ corresponding to the matrices, compare equation (39),

$$
\begin{aligned}
&\mathcal{H}_3 : S^\mu, S^\mu U, S^\mu T S^\nu, S^\mu T S^\nu U, \qquad \mu,\nu = 0,1,2,3,4, \qquad \epsilon := \exp(2\pi \mathrm{i}/5), \\
&S := \begin{bmatrix} \pm\epsilon^3 & 0 \\ 0 & \pm\epsilon^2 \end{bmatrix}, \qquad U := \begin{bmatrix} 0 & \pm 1 \\ \mp 1 & 0 \end{bmatrix}, \qquad T := \frac{1}{\sqrt{5}} \begin{bmatrix} \mp(\epsilon-\epsilon^4) & \pm(\epsilon^2-\epsilon^3) \\ \pm(\epsilon^2-\epsilon^3) & \pm(\epsilon-\epsilon^4) \end{bmatrix}.
\end{aligned} \tag{26}
$$

Hamilton's icosians [6, 7], are the elements of $\mathcal{H}_3$ in the quaternionic basis (equation (4)) of [22] which agrees with [20]:

$$
\begin{aligned}
&\mathcal{H}_3 : \tfrac{1}{2}(\pm 1, \pm 1, \pm 1, \pm 1), \\
&(\pm 1, 0, 0, 0) \text{ and all permutations}, \\
&\tfrac{1}{2}(0, \pm 1, \pm(1-\tau), \pm\tau) \text{ and all even permutations}.
\end{aligned} \tag{27}
$$

In terms of the icosahedral group elements lifted to 3-space, the set (equation (26)) places a five-fold axis (equation (27)), a two-fold axis in the direction 3. To relate the two sets, construct the matrix

$$w := \begin{bmatrix} \cos(\beta/2) & \sin(\beta/2) \\ -\sin(\beta/2) & \cos(\beta/2) \end{bmatrix}, \qquad \cos(\beta) = \sqrt{\frac{\tau+2}{5}}, \qquad \sin(\beta) = \sqrt{\frac{3-\tau}{5}}. \tag{28}$$

This matrix has the conjugation property

$$w\left(\sqrt{\frac{3-\tau}{5}}\sigma_1 + \sqrt{\frac{\tau+2}{5}}\sigma_3\right)w^{-1} = \sigma_3, \tag{29}$$

which shows that conjugation with w rotates a twofold into a neighbouring five-fold icosahedral axis. The matrix w (equation (28)) by conjugation relates the sets (equations (26) and (27) to one another.

5. Spherical harmonics on S^3

The irreps of $SO(4, R)$ may be characterized by eigenvalues of Casimir operators. In second order of the Lie group generators there are two independent Casimir operators, namely the two Casimir operators of $SU^l(2, C)$ and $SU^r(2, C)$ whose Lie algebras given in equations (34), (35) commute with one another. To characterize the special irreps of $SO(4, R)$ carried by the spherical harmonics it suffices to use a single second-order Casimir operator. With a shorthand notation $\partial_{y_i} := \frac{\partial}{\partial y_i}$, we can express this second-order Casimir operator of $SO(4, R)$ in terms of the Laplace operator, the dilatation operator and x^2,

$$\begin{aligned}\Lambda^2 &:= \frac{1}{2}\sum_{i,j=0}^{3}\left(x_i\partial_{x_j} - x_j\partial_{x_i}\right)^2 \\ &= x^2\,\nabla^2 - (x\cdot\nabla)((x\cdot\nabla)+2).\end{aligned} \tag{30}$$

We note that the three operators appearing on the right-hand side of equation (30) form the Lie algebra of the symplectic group $Sp(2, R)$.

The spherical harmonics are homogeneous polynomial solutions of degree λ of the Laplace equation. Application of equation (30) to solutions $P(x)$ of the Laplace equation

$$\triangle\, P(x) = 0, \qquad \triangle = \nabla^2, \tag{31}$$

fixes for spherical harmonics

$$P(x) : (x\cdot\nabla)P(x) = \lambda P(x), \qquad \Lambda^2 P(x) = -\lambda(\lambda+2)P(x), \tag{32}$$

the eigenvalue of the second-order Casimir operator (equation (30)) and the corresponding irreps of $SO(4, R)$.

In the complex coordinates (equation (2)), we have

$$(x\cdot\nabla) = z_1\partial_{z_1} + z_2\partial_{z_2} + \overline{z}_1\partial_{\overline{z}_1} + \overline{z}_2\partial_{\overline{z}_2}. \tag{33}$$

The generators of the groups $SU^r(2, C)$, $SU^l(2, C)$ acting from the right and left on $u(z)$ are then found by left and right actions of the Pauli matrices as

$$\begin{aligned} L^r_+ &= \left[z_1\partial_{z_2} - \overline{z}_2\partial_{\overline{z}_1}\right], \\ L^r_- &= \left[z_2\partial_{z_1} - \overline{z}_1\partial_{\overline{z}_2}\right], \\ L^r_3 &= (1/2)\left[z_1\partial_{z_1} - \overline{z}_1\partial_{\overline{z}_1} - z_2\partial_{z_2} + \overline{z}_2\partial_{\overline{z}_2}\right], \end{aligned} \tag{34}$$

$$L^l_+ = \left[-z_2\partial_{\overline{z}_1} + z_1\partial_{\overline{z}_2}\right],$$
$$L^l_- = \left[\overline{z}_2\partial_{z_1} - \overline{z}_1\partial_{z_2}\right], \tag{35}$$
$$L^l_3 = (1/2)\left[z_1\partial_{z_1} + z_2\partial_{z_2} - \overline{z}_1\partial_{\overline{z}_1} - \overline{z}_2\partial_{\overline{z}_2}\right].$$

The left and right generators in equations (34), (35), respectively, commute but have among themselves the standard $SU(2, C)$ commutation relations

$$[L_3, L_\pm] = \pm L_\pm, \qquad [L_+, L_-] = 2L_3. \tag{36}$$

The spherical harmonics are identical to the Wigner representation functions $D^j_{m',m}$ of $SU(2, C)$ [4], pp 53–67, which in turn are equivalent to the Jacobi polynomials. From a generating function [4] (equation (4.14)) they can be written in the notation of equation (2) as homogeneous polynomials in $(z_1, z_2, \overline{z}_1, \overline{z}_2)$ of degree $2j$,

$$\begin{aligned} D^j_{m',m}(z_1, z_2, \overline{z}_1, \overline{z}_2) &= \left[\frac{(j+m')!(j-m')!}{(j+m)!(j-m)!}\right]^{1/2} \\ &\times \sum_\sigma \frac{(j+m)!(j-m)!}{(j+m'-\sigma)!(m-m'+\sigma)!\sigma!(j-m-\sigma)!} \\ &\times (-1)^{m-m'+\sigma} z_1^{j+m'-\sigma}\, \overline{z}_2^{m-m'+\sigma}\, z_2^\sigma\, \overline{z}_1^{j-m-\sigma}, \qquad j = 0, 1/2, 1, 3/2, \ldots. \end{aligned} \tag{37}$$

The particular spherical harmonics with $m' = j$ are given from equation 37 by

$$D^j_{j,m}(z) = \left[\frac{(2j)!}{(j+m)!(j-m)!}\right]^{1/2} (z_1)^{j+m}(z_2)^{j-m}, \tag{38}$$

they are analytic in (z_1, z_2).

Lemma 5 (Spherical harmonics on S^3 are Wigner D-functions)**.** *The Wigner D^j-functions are homogeneous of degree $\lambda = 2j$ and solve the Laplace equation $\Delta D = 0$. The eigenvalues of the operators $\left(L^l_3, L^r_3\right)$ from equations (34), (35) are (m', m). Under $SO(4, R)$, the spherical harmonics transform according to the irreps $D^j \times D^j$ of $SU^l(2, C) \times SU^r(2, C)$.*

Proof. (i) The analytic D^j-functions (equation (38)) of degree $2j$ are easily seen to vanish under the application of the Laplacian Δ. All other D-functions are obtained by the application of Lie generators from equations (34), (35), which commute with Δ, and so they also must fulfil equations (31), (32). (ii) Under the left/right actions $u \to g_l^{-1}ug_r$, the linear decomposition of $D^j\left(g_l^{-1}ug_r\right)$ in terms of $D^j(u)$ yields $D^j\left(g_l^{-1}\right) \times D^j(g_r)$ as coefficients. □

6. Action of $\mathcal{H}_3$ and polynomial invariants of degree $\lambda = 12$

The points of $S^3 \sim SU(2, C)$ are specified by the two complex numbers (z_1, z_2) in the top row of equation (2). The general right action of $SU^r(2, C)$ on these two complex variables (z_1, z_2) of $S^3 \sim SU(2, C)$ and on their tensor products reads

$$(z'_1, z'_2) = (z_1, z_2)\begin{bmatrix} a & b \\ -\overline{b} & \overline{a} \end{bmatrix}, \qquad (\sqrt{2}z'_1\overline{z}'_2, z'_1\overline{z}'_1 - z'_2\overline{z}'_2, \sqrt{2}\overline{z}'_1 z'_2)$$
$$= (\sqrt{2}z_1\overline{z}_2, z_1\overline{z}_1 - z_2\overline{z}_2, \sqrt{2}\overline{z}_1 z_2)\begin{bmatrix} aa & -\sqrt{2}ab & -bb \\ -\sqrt{2}a\overline{b} & a\overline{a} - b\overline{b} & \sqrt{2}\overline{a}b \\ -\overline{bb} & -\sqrt{2}\overline{ab} & \overline{aa} \end{bmatrix}. \tag{39}$$

Equation (39) shows that the polynomials (z_1, z_2) form a basis of the irrep $D^{1/2}$ of $SU^r(2, C)$.

Felix Klein in his monograph [10] on the icosahedral group lets the binary group $\mathcal{H}_3$ with elements (equation (26)) act on two complex variables (z_1, z_2) in line with equation (39). From a linear fractional transform of the complex variables he in [10], pp 32–4, passes to real Cartesian coordinates (ξ, η, ζ). In terms of equation (39), Klein's correspondence may be written as

$$\left(\sqrt{\tfrac{1}{2}}(\xi + \mathrm{i}\eta), \zeta, \sqrt{\tfrac{1}{2}}(\xi - \mathrm{i}\eta)\right) = (\sqrt{2}z_1\overline{z}_2, z_1\overline{z}_1 - z_2\overline{z}_2, \sqrt{2}\overline{z}_1 z_2), \qquad \xi^2 + \eta^2 + \zeta^2 = 1. \tag{40}$$

It is then easily verified from equation (39) that the action of $SU(2, C)$ on (ξ, η, ζ) reproduces the standard group homomorphism $SU(2, C) \to SO(3, R)$.

In the present analysis, from lemma 3 we infer that the group of deck transformations $\mathcal{H}_3$ acts on $S^3 \sim SU(2, C)$ (equation (2)) from the right as

$$(z_1, z_2) \to (z_1, z_2)g_r, \qquad g_r \in \mathcal{H}_3 < SU^r(2, C). \tag{41}$$

In [10], p 56 (equation (55)), Klein derives the homogeneous polynomial

$$f_k(z_1, z_2) := (z_1 z_2)[(z_1)^{10} + 11(z_1)^5(z_2)^5 - (z_2)^{10}]. \tag{42}$$

By construction, Klein's fundamental polynomial f_k (equation (42)) is $\mathcal{H}_3$-invariant or transforms according to the identity irrep D^{α_0} of $\mathcal{H}_3$, is analytic in (z_1, z_2), and forms the starting point of what Klein calls the icosahedral equation. We emphasize that the explicit form and the invariance of equation (42) are valid if the elements of $\mathcal{H}_3$ are taken as in equation (26). Comparing the spherical harmonics equation (37), the polynomial (equation (42)) up to normalization may be written, in anticipation of equations (45), (46), as $D^6_{6,\alpha_0}(z_1, z_2)$, has degree $2j = 12$, $m' = 6$, and is a superposition of spherical harmonics equation (38) with $m = -5, 0, 5$.

It follows from lemma 3 that the left action of $SU^l(2, C)$ on $S^3 \sim SU(2, C)$ and hence its generators commute with the right action of the group of deck transformations $\mathcal{H}_3$. Therefore when we apply powers $(L^l_-)^r$, $r = 0, \ldots, 12$ of the left lowering operator from equation (34) to the invariant equation (42), we obtain altogether 13 invariant polynomials D^6_{m,α_0}, $m = 6, \ldots, -6$. The first seven are listed in table 1.

The polynomials in table 1 may be normalized by introducing the orthonormal basis (equation (37)).

Among these invariant polynomials we look for one that, by the Klein correspondence equation (40), may be expressed in terms of (ξ, η, ζ). This will allow us in the section 8 to pass to an invariant in the enveloping Lie algebra. A necessary and sufficient condition for rewriting a polynomial $P(z_1, z_2, \tilde{z}_1, \tilde{z}_2)$ in terms of the Klein correspondence is that the complex variables (z_1, z_2) and their complex conjugates must appear with equal powers. Expressed in terms of the generator L^l_3 of equation (35), the condition requires the eigenvalue $m' = 0$ of this generator. This condition singles out from table 1 the invariant polynomial D^6_{0,α_0}. Upon introducing the three powers $(z_1\overline{z}_1 - z_2\overline{z}_2)^p$, $p = 2, 4, 6$, this invariant polynomial can be rewritten as

$$\begin{aligned}
&(L^l_-)^6 f_k := 11 \cdot 6!\mathcal{K}', \\
&\mathcal{K}' = -42((z_1\overline{z}_2)^5 + (\overline{z}_1 z_2)^5)(z_1\overline{z}_1 - z_2\overline{z}_2) + (z_1\overline{z}_1 - z_2\overline{z}_2)^6 - 30(z_1\overline{z}_2)(\overline{z}_1 z_2)(z_1\overline{z}_1 - z_2\overline{z}_2)^4 \\
&\qquad + 90(z_1\overline{z}_2)^2(\overline{z}_1 z_2)^2(z_1\overline{z}_1 - z_2\overline{z}_2)^2 - 20(z_1\overline{z}_2)^3(\overline{z}_1 z_2)^3.
\end{aligned} \tag{43}$$

In this invariant polynomial, we introduce the Klein correspondence equation (40) and rewrite it in terms of (ξ, η, ζ). We obtain the invariant homogeneous polynomial of degree 6

$$\begin{aligned}
\mathcal{K}'(\xi, \eta, \zeta) &= -42(1/2)^5((\xi + \mathrm{i}\eta)^5\zeta + \zeta(\xi - \mathrm{i}\eta)^5) + \zeta^6 - 30(1/2)^2(\xi + \mathrm{i}\eta)(\xi - \mathrm{i}\eta)\zeta^4 \\
&\quad + 90(1/2)^4(\xi + \mathrm{i}\eta)^2(\xi - \mathrm{i}\eta)^2\zeta^2 - 20(1/2)^6(\xi + \mathrm{i}\eta)^3(\xi - \mathrm{i}\eta)^3.
\end{aligned} \tag{44}$$

Table 1. $\mathcal{H}_3$-invariant polynomials D^j_{m',α_0} (unnormalized), obtained from Klein's analytic invariant equation (42) by the application of powers $(L^l_-)^r$, $r = 0, \ldots, 6$; $m' = 6 - r$ of the left lowering operator L^l_- from equation (35).

m'	D^6_{m',α_0}
6	$z_1^{11} z_2 + 11 z_1^6 z_2^6 - z_1 z_2^{11}$
5	$z_1^{10}\,[-z_1\bar{z}_1 + 11 z_2\bar{z}_2]$
	$+11\cdot 6\cdot z_1^5 z_2^5\,[-z_1\bar{z}_1 + z_2\bar{z}_2]$
	$-z_2^{10}\,[-11 z_1\bar{z}_1 + z_2\bar{z}_2]$
4	$-22\cdot z_1^9 \bar{z}_2\,[z_1\bar{z}_1 - 5 z_2\bar{z}_2]$
	$+11\cdot 6\cdot z_1^4 z_2^4 [5((z_1\bar{z}_1)^2 + (z_2\bar{z}_2)^2) - 12 z_1\bar{z}_1 z_2\bar{z}_2]$
	$-22\cdot \bar{z}_1 z_2^9\,[5 z_1\bar{z}_1 - z_2\bar{z}_2]$
3	$-11\cdot 30\cdot z_1^8 \bar{z}_2^2\,[z_1\bar{z}_1 - 3 z_2\bar{z}_2]$
	$-11\cdot 60\cdot z_1^3 z_2^3 [2((z_1\bar{z}_1)^3 - (z_2\bar{z}_2)^3) - 9((z_1\bar{z}_1)^2 z_2\bar{z}_2 - z_1\bar{z}_1 (z_2\bar{z}_2)^2)]$
	$+11\cdot 30\cdot \bar{z}_1^2 z_2^8\,[3 z_1\bar{z}_1 - z_2\bar{z}_2]$
2	$-11\cdot 360\cdot z_1^7 \bar{z}_2^3\,[z_1\bar{z}_1 - 2 z_2\bar{z}_2]$
	$+11\cdot 360\cdot z_1^2 z_2^2 [(z_1\bar{z}_1)^4 + (z_2\bar{z}_2)^4 - 8((z_1\bar{z}_1)^3 z_2\bar{z}_2 + z_1\bar{z}_1 (z_2\bar{z}_2)^3)$
	$+15 (z_1\bar{z}_1)^2 (z_2\bar{z}_2)^2]$
	$-11\cdot 360\cdot \bar{z}_1^3 z_2^7\,[2 z_1\bar{z}_1 - z_2\bar{z}_2]$
1	$-11\cdot 720\cdot z_1^6 \bar{z}_2^4\,[5 z_1\bar{z}_1 - 7 z_2\bar{z}_2]$
	$-11\cdot 720\cdot z_1 z_2$
	$[(z_1\bar{z}_1)^5 - (z_2\bar{z}_2)^5 - 15((z_1\bar{z}_1)^4 z_2\bar{z}_2 - z_1\bar{z}_1 (z_2\bar{z}_2)^4) + 50((z_1\bar{z}_1)^3 (z_2\bar{z}_2)^2$
	$-(z_1\bar{z}_1)^2 (z_2\bar{z}_2)^3)]$
	$+11\cdot 720\cdot \bar{z}_1^4 z_2^6\,[7 z_1\bar{z}_1 - z_2\bar{z}_2]$
0	$-11\cdot 42\cdot 720\cdot z_1^5 \bar{z}_2^5\,[z_1\bar{z}_1 - z_2\bar{z}_2]$
	$+11\cdot 720\cdot ((z_1\bar{z}_1)^6 + (z_2\bar{z}_2)^6 - 36((z_1\bar{z}_1)^5 z_2\bar{z}_2 + z_1\bar{z}_1 (z_2\bar{z}_2)^5)$
	$+225\cdot ((z_1\bar{z}_1)^4 (z_2\bar{z}_2)^2 + (z_1\bar{z}_1)^2 (z_2\bar{z}_2)^4) - 400\cdot (z_1\bar{z}_1)^3 (z_2\bar{z}_2)^3)$
	$-11\cdot 42\cdot 720\cdot \bar{z}_1^5 z_2^5\,[z_1\bar{z}_1 - z_2\bar{z}_2]$

Lemma 6 (The polynomial equation (44) is an $\mathcal{H}_3$-invariant of degree 6 in (ξ, η, ζ))**.** *The polynomial $\mathcal{K}'$ equation (43) by construction is invariant under the binary icosahedral group $\mathcal{H}_3$ acting on the complex coordinates. Moreover, the real coordinates (ξ, η, ζ) (equation (40)) carry a standard three-dimensional orthogonal irrep of the icosahedral group (five-fold axis along 3-axis). Therefore the polynomial equation (44) is also an invariant of degree 6 in (ξ, η, ζ) with respect to this irrep.*

7. Subduction of irreps for the pair $SO(4, R) > \mathcal{H}_3$

The group of deck transformations in the geometry of section 2 acts on $S^3 \sim SU(2, C)$ as the binary icosahedral group $\mathcal{H}_3$ from the right. We wish to find on S^3 the homogeneous solutions of the Laplace equation (31) which belong to the identity irrep D^{α_0} of $\mathcal{H}_3$. This amounts to decomposing given irreps of a group G into irreps of a subgroup $G > H$, a process called subduction of irreps and reciprocal to induction. Actually, we shall subduce the bases of irreps rather than the irreps themselves. For the pair $SO(4, R) > \mathcal{H}_3$, by lemma 2 we can refine this analysis to $SO(4, R) > SU^r(2, C) > \mathcal{H}_3$. The fixed irreps $D^j \times D^j$ of

$SO(4, R)$ subduce the single irrep D^j of $SU^r(2, C)$, and so it remains to subduce the irreps in $SU^r(2, C) > \mathcal{H}_3$. This subduction has been studied by Cesare and Del Duca [3]. They distinguish between fermionic and bosonic irreps of $\mathcal{H}_3$, corresponding to odd and even values of $\lambda = 2j$. They give recursive expressions for the decomposition under $\mathcal{H}_3$ of all irreps of $SU(2, C)$ and provide corresponding projection operators.

In the following sections, we shall develop from Klein's fundamental invariant equation (42) an alternative and powerful operator tool for subducing and quantizing an orthonormal and complete basis of eigenstates on M.

For the full subduction $SO(4, R) > \mathcal{H}_3$ we recall that $\mathcal{H}_3$ commutes with $SU^l(2, C)$ so that we are free to choose the only free representation label m' of the latter group. For $m' = j$, which corresponds to the simple analytic spherical harmonics equation (38), or for any $m' : -j \leqslant m' \leqslant j$, the subduction of the basis of spherical harmonics equation (38) reads

$$D^j_{j,\alpha} = \sum_m D^j_{j,m}(z, \overline{z})c_{m,\alpha}, \qquad -j \leqslant m' \leqslant j, \tag{45}$$

$$D^j_{m',\alpha} = \sum_m D^j_{m',m}(z, \overline{z})c_{m,\alpha}, \qquad -j \leqslant m' \leqslant j. \tag{46}$$

The coefficients $c_{m,\alpha}$ are yet to be determined but must be independent of m', since we could pass from equation (45) to equation (46) by the application of the left lowering operator L^l_-. This procedure, exemplified by table 1 for $\alpha = \alpha_0$, reduces the search for spherical harmonics invariant under $\mathcal{H}_3$ by a factor $(2j + 1)$.

Lemma 7 (Classification of $\mathcal{H}_3$-invariant polynomials by eigenvalues of L^l_3)**.** *For given $\lambda = 2j$, the spherical harmonics which belong to a fixed irrep of $\mathcal{H}_3$ can be grouped into sets (equation (46)) whose $(2j + 1)$ members are orthogonal and distinguished by the eigenvalues m', $-j \leqslant m' \leqslant j$ of L^l_3.*

8. Irreducible $\mathcal{H}_3$ states and the generalized Casimir operator $\mathcal{K}$ for $SU(2, C) > \mathcal{H}_3$

Casimir operators like Λ^2 are used for fixing the irreps of a given Lie group. To distinguish and label the subduction of irreps in $G > H$, we follow the paradigm given by Bargmann and Moshinsky [1] and construct in the enveloping algebra of $SO(4, R)$ a generalized Casimir operator $\mathcal{C}$ associated to the group/subgroup pair $G > H$. This Lie algebraic operator technique was applied in [1] to $SU(3, C) > SO(3, R)$ and in [14], pp 263–8, to the continuous/discrete pair $O(3, R) > D^{[3,1]}(S_4)$.

A generalized Casimir operator must have the following properties:

(i) $\mathcal{C}$ must commute with the Casimir operators of the group G and
(ii) $\mathcal{C}$ must be Hermitian and invariant under the subgroup H but not under the full group G.

Once we have found this operator for the groups $SU^r(2, C) > \mathcal{H}_3$, by standard symmetry arguments its modes transforming with any fixed irrep of $SU(2, C)$ must fall into subsets of degenerate eigenstates transforming with fixed irreps D^α of $\mathcal{H}_3$.

Condition (i) is fulfilled by any operator-valued polynomial $\mathcal{P}$ in the generators of $SU^r(2, C)$. The set of these polynomials forms the enveloping algebra of $SU^r(2, C)$. The generators when taken as $\left(L^r_1, L^r_2, L^r_3\right)$ transform linearly according to equation (39) under $SU^r(2, C)$ as the vector (ξ, η, ζ).

Condition (ii) requires the polynomial to be invariant under $\mathcal{H}_3$. We would like to get invariance by substituting $\left(L^r_1, L^r_2, L^r_3\right)$ in the invariant polynomial $\mathcal{K}'$ of degree 6 from equation (44). But, as these generators do not commute, a naive substitution of L-components into the polynomial equation (44) of degree 6 does not guarantee the invariant transformation property under $\mathcal{H}_3$.

The transformation property for any operator-valued polynomial $\mathcal{P}(A_1, A_2, \ldots)$ is maintained if, after naive substitution, we symmetrize it by adding all polynomials obtained from any permutation of the operators $(A_1, A_2, \ldots)$ involved, and dividing by the number of permutations. The abelianization of this symmetrized operator-valued polynomial clearly would reconstruct the polynomial in commuting vector components and its transformation property. We denote the operation of symmetrization by the symbol $\mathrm{Sym}(\mathcal{P})$.

Lemma 8 (The $\mathcal{H}_3$-invariant operator $\mathcal{K}$)**.** *The generalized Casimir operator $\mathcal{K}$ for the group/subgroup chain $SU^r(2, C) > \mathcal{H}_3$ is the Hermitian polynomial operator*

$$\begin{aligned}\mathcal{K}\big(L_1^r, L_2^r, L_3^r\big) &:= \mathrm{Sym}\big(\mathcal{K}'_{(\xi,\eta,\zeta)\to(L_1^r,L_2^r,L_3^r)}\big) = -42(1/2)^5\,\mathrm{Sym}\big(L_+^5 L_3 + L_3 L_-^5\big)\\ &\quad + \mathrm{Sym}\big(L_3^6\big) - 30(1/2)^2\,\mathrm{Sym}\big(L_+ L_- L_3^4\big) + 90(1/2)^4\,\mathrm{Sym}\big(L_+^2 L_-^2 L_3^2\big)\\ &\quad - 20(1/2)^6\,\mathrm{Sym}\big(L_+^3 L_-^3\big). \end{aligned} \tag{47}$$

On the right-hand side of equation (47) and in what follows we drop for simplicity the upper index for the right action. The Hermitian property is either manifest or obtained upon symmetrization. We observe that, in the standard $|jm\rangle$ basis, the only off-diagonal terms of $\mathcal{K}$ are the first two. The other four terms of $\mathcal{K}$ can be expressed as polynomials in L^2, L_3 and so are diagonal. Algebraic expressions for the operations Sym *of symmetrizations in equation (47) are given in the appendix.*

9. The spectrum of $\mathcal{K}$ and the quantization of M by its eigenstates

To see if the operator $\mathcal{K}$ completely resolves the subduction of irreps in $SO(4, R) > \mathcal{H}_3$ we must explore its eigenvalues and eigenstates. Since $\mathcal{K}$ is $\mathcal{H}_3$-invariant, its eigenstates on $\tilde{M} = S^3$ carry irreps D^α of $\mathcal{H}_3$. The operator $\mathcal{K}$ classifies and by its eigenvalues quantizes the irrep D^α of $\mathcal{H}_3$ for fixed j on S^3. The states in general live on the universal cover $\tilde{M} = S^3$ and must have an additional degeneracy corresponding to the dimension $|\alpha|$ of the irrep D^α. The combined eigenspaces of $\mathcal{K}$ span the same linear space as the spherical harmonics on S^3, but now transforming under irreps D^α of $\mathcal{H}_3$.

To diagonalize $\mathcal{K}$ we apply lemma 5 in eigenspaces of fixed $\lambda = 2j$ and for $m' = j$. The corresponding subspace $\mathcal{L}_j^j$ has dimension $2j + 1$ and an analytic basis $|jm\rangle$ equation (38) w.r.t. the right action of $SU^r(2, C)$. The only off-diagonal elements arise from the first two terms of $\mathcal{K}$ equation (47), while the other ones are diagonal in this scheme.

We now take full advantage of Klein's expressions equation (26) for the elements of $\mathcal{H}_3$. Since the off-diagonal part of $\mathcal{K}$ links the basis states modulo 5, we split the values of $m, -j \leqslant m \leqslant j$ as $m \equiv \mu$ modulo 5 and the subspace $\mathcal{L}_j^j$ into orthogonal subspaces $\mathcal{L}_{j,\mu}^j$ of fixed μ. Within a subspace $\mathcal{L}_{j,\mu}^j$, the matrix of $\mathcal{K}$ is tridiagonal; moreover, with non-zero off-diagonal entries. From these properties it is easily shown [1] that, within $\mathcal{L}_{j,\mu}^j$, the spectrum of $\mathcal{K}$ is non-degenerate. This implies that the $|\alpha|$ degenerate eigenstates of $\mathcal{K}$ belonging to the same irrep D^α are completely distinguished by the index μ. The other degeneracy of $\mathcal{K}$, resulting from its commuting with $SU^l(2, C)$, is resolved by diagonalization of L_3^l with eigenvalue m', as exemplified in table 1.

Among the irrep D^α modes on $\tilde{M} = S^3$ is the subset of proper eigenstates for the topological Poincaré 3-manifold M. They belong exclusively to the identity irrep of $\mathcal{H}_3$, which we denote by $D^{\alpha_0} \equiv 1$. Since any value taken by such a polynomial function on S^3 is repeated on all copies of M under $\mathcal{H}_3$, the domain of these states can be uniquely restricted from $\tilde{M}$ to the Poincaré 3-manifold M. For these invariant eigenstates we get sharper selection rules of the subspaces $\mathcal{L}^j_{j,\mu}$. We have taken the elements of $\mathcal{H}_3$ in the setting due to Klein [10] (equation (26)). The binary preimage of the icosahedral five-fold rotation around the 3-axis is generated by the element S from this equation. For the eigenstates belonging to D^{α_0}, invariance in particular under the element $S \in \mathcal{H}_3$ implies that they can occur only in the subspaces $\left(\mathcal{L}^j_{m',\mu}, \mu = 0\right)$. These subspaces appear only for $2j =$ even, and so there can be no $\mathcal{H}_3$-invariant eigenmodes of M with $2j =$ odd. Due to the non-degeneracy of $\mathcal{K}$ on these subspaces, any two $\mathcal{H}_3$-invariant eigenstates on the same subspace $\left(\mathcal{L}^j_{m',\mu}, \mu = 0\right)$ must differ in their eigenvalues.

Theorem 1 (The $\mathcal{H}_3$-invariant operator $\mathcal{K}$ equation (47) quantizes the Poincaré's dodecahedral 3-manifold M)**.** *A complete set of invariant eigenmodes on the dodecahedral Poincaré topological manifold M is given by those eigenstates with eigenvalue κ of $\mathcal{K}$ which belong to the identity irrep D^{α_0} of $\mathcal{H}_3$. These eigenstates occur only in the orthogonal subspaces $\left(\mathcal{L}^j_{m',\mu}, \mu = 0, 2j = \text{even}, -j \leqslant m' \leqslant j\right)$. For fixed j, they are of degree $2j$ and are quantized by the eigenvalue κ of $\mathcal{K}$. In the subspace $\left(\mathcal{L}^j_{(m',\mu)}, m' = j, \mu = 0\right)$, the eigenstates are non-degenerate and are homogeneous polynomials analytic in (z_1, z_2) as in equation (38) and similar to Klein's f_k equation (42). The $2j$ partners with the same eigenvalue κ and eigenvalue $m' = j-1, \ldots, -j$ of L^l_3 are obtained by the multiple application of the lowering operator L^l_-, as given in table 1.*

Theorem 1 verifies the preview given in the introduction. The different topologies of $\tilde{M} = S^3$ versus M give rise to different eigenmodes: on $\tilde{M}$, the eigenmodes are all the spherical harmonics of equation (37). To pass to the eigenmodes of M, one must select from them the $\mathcal{H}_3$-invariant ones, which have a unique restriction to M. The number of invariant eigenmodes for fixed j can also be found from [3], the eigenmodes become eigenstates of $\mathcal{K}$.

Lemma 9. *No eigenmodes of M exist for $\lambda = 2j < 12$.*

Proof. The irreps D^j, $j = (1, 2)$ of $SU^r(2)$ remain irreducible when subduced to $\mathcal{H}_3$ [3], p 5, and so cannot yield invariant eigenmodes. For $j = (3, 4, 5)$, the diagonalization of $\mathcal{K}$ in section 10 shows only degenerate and no invariant eigenstates at all. □

10. Diagonalization of $\mathcal{K}$ for the subspaces $\mathcal{L}^j$, $j = 1, \ldots, 6$

First of all, we observe that in the subspaces $\mathcal{L}^j$, $j = 1, 2$, the operator $\mathcal{K}$ gives vanishing results. The reason is as follows: the polynomials $\mathcal{K}'$ (equation (43)) as well as f_k under $SU^r(2, C)$ transform according to the irrep D^j, $j = 6$. It then follows from the construction (equation (45)) that the operator $\mathcal{K}$ under $SU^r(2, C)$ transforms as part of a tensor operator $\mathcal{K} = \mathcal{K}^q$ of rank $q = 6$. From standard selection rules for tensor operators, non-vanishing of its matrix elements $\langle j_2 m_2 | \mathcal{K}^6 | j_1 m_1 \rangle$ requires $j_1 + j_2 \geqslant 6 \geqslant |j_1 - j_2|$ which excludes $j_1 = j_2 = (1, 2)$. The irreps of $\mathcal{H}_3$ for $j = (1, 2)$ can be analysed independent of $\mathcal{K}$ but provide no invariant modes, see lemma 9.

As an illustration of $\mathcal{K}$ and its diagonalization we consider the subspaces $\mathcal{L}^j$, $j = 3, 4, 5, 6$. Within each subspace $\mathcal{L}^j_{j,\mu}$, the submatrix $\mathcal{K}^\mu$ of $\mathcal{K}$, the diagonalizing matrix V^μ and the diagonal form $\mathcal{K}^{\mu,\text{diag}}$ fulfil

$$\mathcal{K}^\mu \cdot V^\mu = V^\mu \cdot \mathcal{K}^{\mu,\text{diag}}. \tag{48}$$

Table 2. Eigenvalues κ of $\mathcal{K}$ and their multiplicities $|\alpha|$ in the form $\kappa^{|\alpha|}$ for $j = 3, 4, 5, 6$.

j	$\kappa^{\vert\alpha\vert}$
3	$(-225)^3, \left(\frac{675}{4}\right)^4$
4	$\left(\frac{7875}{4}\right)^4, (-1575)^5$
5	$\left(-\frac{23\,625}{2}\right)^3, (7875)^3, \left(\frac{4725}{2}\right)^5$
6	$(-51\,975)^1, \left(-\frac{51\,975}{2}\right)^3, (23\,625)^4, \left(\frac{14\,175}{2}\right)^5$

As a survey we give in table 2 the eigenvalues κ and multiplicities $|\alpha|$ in the form $\kappa^{|\alpha|}$ as a function of j.

We give the submatrices of $\mathcal{K}$ according to equation (48) in closed algebraic form in equations (49)–(52) for $j = 3, 4, 5, 6$, respectively. The extension of the diagonalization to $j > 6$ presents no problem.

$$\boldsymbol{j = 3}, \qquad \kappa^{|\alpha|} = (-225)^3, \left(\frac{675}{4}\right)^4 \tag{49}$$

$\mu = 0 : m = 0$:

$$\mathcal{K}^0 = [-225], \qquad V^0 = [1], \qquad \mathcal{K}^{0,\mathrm{diag}} = [-225]$$

$\mu = 1 : m = 1$

$$\mathcal{K}^1 = \left[\frac{675}{4}\right], \qquad V^1 = [1], \qquad \mathcal{K}^{1,\mathrm{diag}} = \left[\frac{675}{4}\right]$$

$\mu = 2 : m = (-3, 2)$:

$$\mathcal{K}^2 = \begin{bmatrix} \frac{45}{4} & 315\sqrt{\frac{3}{8}} \\ 315\sqrt{\frac{3}{8}} & -\frac{135}{2} \end{bmatrix},$$

$$V^2 = \begin{bmatrix} -\sqrt{\frac{2}{5}} & \sqrt{\frac{3}{5}} \\ \sqrt{\frac{3}{5}} & \sqrt{\frac{2}{5}} \end{bmatrix}, \qquad \mathcal{K}^{2,\mathrm{diag}} = \begin{bmatrix} -225 & \\ & \frac{675}{4} \end{bmatrix},$$

$\mu = 3 : m = (-2, 3)$:

$$\mathcal{K}^3 = \begin{bmatrix} -\frac{135}{2} & -315\sqrt{\frac{3}{8}} \\ -315\sqrt{\frac{3}{8}} & \frac{45}{4} \end{bmatrix},$$

$$V^3 = \begin{bmatrix} \sqrt{\frac{3}{5}} & -\sqrt{\frac{2}{5}} \\ \sqrt{\frac{2}{5}} & \sqrt{\frac{3}{5}} \end{bmatrix}, \qquad \mathcal{K}^{3,\mathrm{diag}} = \begin{bmatrix} -225 & \\ & \frac{675}{4} \end{bmatrix},$$

$\mu = 4 : m = -1$

$$\mathcal{K}^4 = \left[\frac{675}{4}\right], \qquad V^4 = [1], \qquad \mathcal{K}^{4,\mathrm{diag}} = \left[\frac{675}{4}\right].$$

$$\boldsymbol{j=4}, \qquad \kappa^{|\alpha|} = \left(\frac{7875}{4}\right)^4, (-1575)^5 \tag{50}$$

$\mu = 0 : m = 0$:

$$\mathcal{K}^0 = [-1575], \qquad V^0 = [1], \qquad \mathcal{K}^{0,\mathrm{diag}} = [-1575]$$

$\mu = 1 : m = (-4, 1)$:

$$\mathcal{K}^1 = \begin{bmatrix} 315 & 945\sqrt{\frac{7}{2}} \\ 945\sqrt{\frac{7}{2}} & \frac{315}{4} \end{bmatrix},$$

$$V^1 = \begin{bmatrix} 2\sqrt{\frac{2}{15}} & -\sqrt{\frac{7}{15}} \\ \sqrt{\frac{7}{15}} & 2\sqrt{\frac{2}{15}} \end{bmatrix}, \qquad \mathcal{K}^{1,\mathrm{diag}} = \begin{bmatrix} \frac{7875}{4} & \\ & -1575 \end{bmatrix},$$

$\mu = 2 : m = (-3, 2)$:

$$\mathcal{K}^2 = \begin{bmatrix} -\frac{5355}{4} & 945\sqrt{\frac{7}{8}} \\ 945\sqrt{\frac{7}{8}} & \frac{3465}{2} \end{bmatrix},$$

$$V^2 = \begin{bmatrix} \sqrt{\frac{1}{15}} & -\sqrt{\frac{14}{15}} \\ \sqrt{\frac{14}{15}} & \sqrt{\frac{1}{15}} \end{bmatrix}, \qquad \mathcal{K}^{2,\mathrm{diag}} = \begin{bmatrix} \frac{7875}{4} & \\ & -1575 \end{bmatrix},$$

$\mu = 3 : m = (-2, 3)$:

$$\mathcal{K}^3 = \begin{bmatrix} \frac{3465}{2} & -945\sqrt{\frac{7}{8}} \\ -945\sqrt{\frac{7}{8}} & -\frac{5355}{4} \end{bmatrix},$$

$$V^3 = \begin{bmatrix} -\sqrt{\frac{14}{15}} & \sqrt{\frac{1}{15}} \\ \sqrt{\frac{1}{15}} & \sqrt{\frac{14}{15}} \end{bmatrix}, \qquad \mathcal{K}^{3,\mathrm{diag}} = \begin{bmatrix} \frac{7875}{4} & \\ & -1575 \end{bmatrix},$$

$\mu = 4 : m = (-1, 4)$:

$$\mathcal{K}^4 = \begin{bmatrix} \frac{315}{4} & -945\sqrt{\frac{7}{2}} \\ -945\sqrt{\frac{7}{2}} & 315 \end{bmatrix},$$

$$V^4 = \begin{bmatrix} -\sqrt{\frac{7}{15}} & \sqrt{\frac{8}{15}} \\ \sqrt{\frac{8}{15}} & \sqrt{\frac{7}{15}} \end{bmatrix}, \qquad \mathcal{K}^{4,\mathrm{diag}} = \begin{bmatrix} \frac{7875}{4} & \\ & -1575 \end{bmatrix}.$$

$$\boldsymbol{j=5}, \qquad \kappa^{|\alpha|} = \left(-\frac{23\,625}{2}\right)^3, (7875)^3, \left(\frac{4725}{2}\right)^5 \tag{51}$$

$\mu = 0 : m = (-5, 0, 5)$:

$$\mathcal{K}^0 = \begin{bmatrix} \frac{4725}{2} & \frac{4725\sqrt{7}}{2} & 0 \\ \frac{4725\sqrt{7}}{2} & -6300 & -\frac{4725\sqrt{7}}{2} \\ 0 & -\frac{4725\sqrt{7}}{2} & \frac{4725}{2} \end{bmatrix},$$

$$V^0 = \begin{bmatrix} -\sqrt{\frac{7}{50}} & -\frac{3}{5} & \sqrt{\frac{1}{2}} \\ \sqrt{\frac{36}{50}} & -\frac{\sqrt{7}}{5} & 0 \\ \sqrt{\frac{7}{50}} & \frac{3}{5} & \sqrt{\frac{1}{2}} \end{bmatrix}, \qquad \mathcal{K}^{0,\mathrm{diag}} = \begin{bmatrix} -\frac{23\,625}{2} & & \\ & 7875 & \\ & & \frac{4725}{2} \end{bmatrix},$$

$\mu = 1 : m = (-4, 1)$:

$$\mathcal{K}^1 = \begin{bmatrix} -7560 & \frac{2835\sqrt{21}}{2} \\ \frac{2835\sqrt{21}}{2} & -1890 \end{bmatrix},$$

$$V^1 = \begin{bmatrix} -\sqrt{\frac{7}{10}} & \sqrt{\frac{3}{10}} \\ \sqrt{\frac{3}{10}} & \sqrt{\frac{7}{10}} \end{bmatrix}, \qquad \mathcal{K}^{1,\mathrm{diag}} = \begin{bmatrix} -\frac{23\,625}{2} & \\ & \frac{4725}{2} \end{bmatrix},$$

$\mu = 2 : m = (-3, 2)$:

$$\mathcal{K}^2 = \begin{bmatrix} \frac{9135}{2} & 2205\sqrt{\frac{3}{2}} \\ 2205\sqrt{\frac{3}{2}} & 5670 \end{bmatrix},$$

$$V^2 = \begin{bmatrix} \sqrt{\frac{2}{5}} & -\sqrt{\frac{3}{5}} \\ \sqrt{\frac{3}{5}} & \sqrt{\frac{2}{5}} \end{bmatrix}, \qquad \mathcal{K}^{2,\mathrm{diag}} = \begin{bmatrix} 7875 & \\ & \frac{4725}{2} \end{bmatrix},$$

$\mu = 3 : m = (-2, 3)$:

$$\mathcal{K}^3 = \begin{bmatrix} 5670 & -2205\sqrt{\frac{3}{2}} \\ -2205\sqrt{\frac{3}{2}} & \frac{9135}{2} \end{bmatrix},$$

$$V^3 = \begin{bmatrix} -\sqrt{\frac{3}{5}} & \sqrt{\frac{2}{5}} \\ \sqrt{\frac{2}{5}} & \sqrt{\frac{3}{5}} \end{bmatrix}, \qquad \mathcal{K}^{3,\mathrm{diag}} = \begin{bmatrix} 7875 & \\ & \frac{4725}{2} \end{bmatrix},$$

$\mu = 4 : m = (-1, 4)$:

$$\mathcal{K}^4 = \begin{bmatrix} -1890 & -\frac{2835\sqrt{21}}{2} \\ -\frac{2835\sqrt{21}}{2} & -7560 \end{bmatrix},$$

$$V^4 = \begin{bmatrix} \sqrt{\frac{3}{10}} & -\sqrt{\frac{7}{10}} \\ \sqrt{\frac{7}{10}} & \sqrt{\frac{3}{10}} \end{bmatrix}, \qquad \mathcal{K}^{4,\mathrm{diag}} = \begin{bmatrix} -\frac{23\,625}{2} & \\ & \frac{4725}{2} \end{bmatrix},$$

$$\boldsymbol{j=6}, \qquad \kappa^{|\alpha|} = (-51\,975)^1, \left(-\frac{51\,975}{2}\right)^3, (23\,625)^4, \left(\frac{14\,175}{2}\right)^5. \qquad (52)$$

$\mu = 0, m = (-5, 0, 5)$:

$$\mathcal{K}^0 = \begin{bmatrix} -\frac{51\,975}{2} & \frac{4725\sqrt{77}}{2} & 0 \\ \frac{4725\sqrt{77}}{2} & -18\,900 & -\frac{4725\sqrt{77}}{2} \\ 0 & -\frac{4725\sqrt{77}}{2} & -\frac{51975}{2} \end{bmatrix},$$

$$V^0 = \begin{bmatrix} -\sqrt{\frac{7}{25}} & \sqrt{\frac{1}{2}} & -\sqrt{\frac{11}{50}} \\ \sqrt{\frac{11}{25}} & 0 & -2\sqrt{\frac{7}{50}} \\ \sqrt{\frac{7}{25}} & \sqrt{\frac{1}{2}} & \sqrt{\frac{11}{50}} \end{bmatrix}, \quad \mathcal{K}^{0,\mathrm{diag}} = \begin{bmatrix} -51\,975 & & \\ & -\frac{51\,975}{2} & \\ & & \frac{14\,175}{2} \end{bmatrix}.$$

$\mu = 1, m = (-4, 1, 6)$:

$$\mathcal{K}^1 = \begin{bmatrix} 3780 & \frac{19\,845\sqrt{3}}{2} & 0 \\ \frac{19\,845\sqrt{3}}{2} & -9450 & -6615\sqrt{\frac{11}{2}} \\ 0 & -6615\sqrt{\frac{11}{2}} & 10\,395 \end{bmatrix},$$

$$V^1 = \begin{bmatrix} -\sqrt{\frac{11}{50}} & -\sqrt{\frac{6}{25}} & 3\sqrt{\frac{3}{50}} \\ \sqrt{\frac{33}{50}} & -2\sqrt{\frac{2}{25}} & \sqrt{\frac{1}{50}} \\ \sqrt{\frac{6}{50}} & \sqrt{\frac{11}{25}} & \sqrt{\frac{22}{50}} \end{bmatrix}, \quad \mathcal{K}^{1,\mathrm{diag}} = \begin{bmatrix} -\frac{51\,975}{2} & & \\ & 23\,625 & \\ & & \frac{14\,175}{2} \end{bmatrix}.$$

$\mu = 2, m = (-3, 2)$:

$$\mathcal{K}^2 = \begin{bmatrix} \frac{40\,635}{2} & 6615 \\ 6615 & 10\,395 \end{bmatrix},$$

$$V^2 = \begin{bmatrix} 2\sqrt{\frac{1}{5}} & -\sqrt{\frac{1}{5}} \\ \sqrt{\frac{1}{5}} & 2\sqrt{\frac{1}{5}} \end{bmatrix}, \qquad \mathcal{K}^{2,\mathrm{diag}} = \begin{bmatrix} 23\,625 & \\ & \frac{14175}{2} \end{bmatrix}$$

$\mu = 3, m = (-2, 3)$:

$$\mathcal{K}^3 = \begin{bmatrix} 10\,395 & -6615 \\ -6615 & \frac{40\,635}{2} \end{bmatrix},$$

$$V^3 = \begin{bmatrix} -\sqrt{\frac{1}{5}} & 2\sqrt{\frac{1}{5}} \\ 2\sqrt{\frac{1}{5}} & \sqrt{\frac{1}{5}} \end{bmatrix}, \qquad \mathcal{K}^{3,\mathrm{diag}} = \begin{bmatrix} 23\,625 & \\ & \frac{14\,175}{2} \end{bmatrix}.$$

$\mu = 4, m = (-6, -1, 4)$:

$$\mathcal{K}^4 = \begin{bmatrix} 10\,395 & 6615\sqrt{\frac{11}{2}} & 0 \\ 6615\sqrt{\frac{11}{2}} & -9450 & -\frac{19\,845\sqrt{3}}{2} \\ 0 & -\frac{19\,845\sqrt{3}}{2} & 3780 \end{bmatrix},$$

$$V^4 = \begin{bmatrix} -\sqrt{\frac{6}{50}} & -\sqrt{\frac{11}{25}} & \sqrt{\frac{22}{50}} \\ \sqrt{\frac{33}{50}} & -2\sqrt{\frac{2}{25}} & -\sqrt{\frac{1}{50}} \\ \sqrt{\frac{11}{50}} & \sqrt{\frac{6}{25}} & \sqrt{\frac{27}{50}} \end{bmatrix}, \quad \mathcal{K}^{4,\mathrm{diag}} = \begin{bmatrix} -\frac{51\,975}{2} & & \\ & 23\,625 & \\ & & \frac{14\,175}{2} \end{bmatrix}$$

The structures in equations (49)–(52) are algebraic and display the typical properties of the spectrum and eigenstates of $\mathcal{K}$ as described in theorem 1: within each subspace $\mathcal{L}^j_\mu$ there are no degenerate eigenvalues. Eigenvalues are repeated only in subspaces $\mu' \neq \mu$. The only single eigenvalue $\kappa = -51\,975$ occurs in $\mathcal{L}^6_{6,0}$ and determines a single eigenstate on the Poincaré manifold. The corresponding eigenstate when written in terms of the normalized polynomials (equation (38)) as

$$\left[\frac{z_1 z_2^{11}}{\sqrt{11!}} \cdot \left(-\sqrt{\frac{7}{25}}\right) + \frac{z_1^6 z_2^6}{\sqrt{6!6!}} \cdot \sqrt{\frac{11}{25}} + \frac{z_1^{11} z_2}{\sqrt{11!}} \cdot \sqrt{\frac{7}{25}}\right] \equiv -f_k(z_1, z_2)\sqrt{\frac{7}{25 \cdot 11!}}, \tag{53}$$

proves to be proportional to Klein's analytic invariant (equation (42)). So the invariant operator $\mathcal{K}$ (equation (47)), derived from Klein's invariant, reproduces this invariant as an eigenstate quantized by its eigenvalue. This crucial result confirms the consistency of the present approach. Moreover, the states given in table 1 up to normalization are (part of) the 12 m'-partners of Klein's invariant and belong to the same eigenvalue of $\mathcal{K}$. These form the lowest degree eigenmodes of M.

11. Discussion

We compare the analysis with recent work on cosmic topology. Multiply connected topologies for cosmology have become a field of intense study [15, 18]. The authors of [19, 29] propose in particular the Poincaré 3-manifold M as a candidate for the space part of the cosmos. In their terminology, it belongs to the single-action manifolds, corresponding to the right action of the group of deck transformations. In lemma 2 we prove this right action from the gluing prescription of [26, 27] for the dodecahedral 3-manifold.

With the goal of expanding the temperature fluctuations of the cosmic microwave background (CMB), the eigenmodes of M and of similar 3-manifolds are studied in [17] by a ghost, an averaging, and by a projection method.

There are some conceptual differences: the authors of [17], p 4687, speak of eigenstates of the Laplacian, whereas Weeks [29], p 615, characterizes the modes as homogeneous harmonic polynomials of degree k solving the Laplace equation, $\Delta P = 0$. The latter notion agrees with the spherical harmonics according to section 5, which by equations (30), (31) and (32) diagonalize the Casimir operator of $SO(4, R)$, with $\lambda = 2j$ playing the role of k. The ghost method of [17] looks for a restriction of eigenmodes of the universal covering S^3 to those of M, which agrees with the reasoning given in section 1, but no general expressions for the eigenmodes are given. Weeks, [29], p 615, points out the need for an accurate and efficient computation of the eigenmodes. These properties are provided by the present operator and quantization method.

Lachièze-Rey [16] discusses the eigenmodes of M in terms of modified spherical harmonics on S^3. His basis explicitly reduces the cyclic group Z_5 as in Klein's analysis. The results on eigenmodes in [16] table 1 are given in numerical and not in algebraic form.

The selection rules of eigenmodes of M versus those of $\tilde{M} = S^3$ are emphasized by Weeks [29]. However, dodecahedral quadrupole and octupole modes $l = 2, 3$, as discussed in [19, 29], are in conflict with the selection rule $k = 2j \geqslant 12$ on M. All the selection rules for eigenmodes can be read off already from the irrep subduction rules for $SU(2, C) > \mathcal{H}_3$ given in [3], complemented by the multiplicity $(2j + 1)$ arising from $\mathcal{H}_3$ commuting with $SU^l(2, C)$. Ikeda [9] gives the lowest degree of a non-vanishing eigenmode of M as $k = 12$ with multiplicity 13. These values agree with $j = 6$ and multiplicity $2j + 1 = 13$ of the present analysis.

12. Conclusion

The eigenmodes of the Poincaré dodecahedral topological 3-manifold M are characterized by Lie algebraic operator techniques as eigenstates with eigenvalues. Guided by homotopy, the group of deck transformations and related Coxeter groups, by F Klein's fundamental invariant polynomial and by representation theory, a Hermitian generalized Casimir operator $\mathcal{K}$ for the group/subgroup subduction $SO(4, R) > SU^r(2, C) > \mathcal{H}_3$ is constructed. Its eigenstates are obtained from homogeneous polynomials, analytic in two complex variables (z_1, z_2). The degeneracies in the spectrum of $\mathcal{K}$ are completely resolved. The proper selection rules for passing from eigenstates on the universal covering $\tilde{M} = S^3$ to eigenstates on M arise from the spectrum of $\mathcal{K}$. The basis of eigenstates of $\mathcal{K}$ is well suited for the expansion of observables like the temperature fluctuation of the CMB.

The present Lie algebraic operator techniques from representation theory are not restricted to M, they can be developed for other models and groups [17] considered in cosmic topology. For example, one could think of the orbifold associated with the Coxeter group (equation (14)) consisting of the fundamental simplex for this group described in section 4.

Hyperbolic counterparts of the Poincaré dodecahedral 3-manifold are the Weber–Seifert dodecahedral 3-manifold [26] and variants of it given by Best [2]. Similar methods from group theory, including a hyperbolic Coxeter group, apply to the Weber–Seifert 3-manifold, compare [13].

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Appendix. Explicit symmetrization of the operator $\mathcal{K}$

The operation Sym of symmetrization of an operator-valued polynomial is well defined, it can be used to compute the matrix, eigenvectors and eigenvalues of $\mathcal{K}$ (equation (47)). Here, we develop an alternative efficient method for obtaining its matrix in the $|jm\rangle$ scheme, based on the representation of the Lie algebra of $SU^r(2, C)$ and its commutators (equation (36)).

We begin with the first two terms in the H_3-invariant operator $\mathcal{K}$ (equation (47)) and obtain with the help of the commutators

$$\mathrm{Sym}\big(L_+^5 L_3\big) = \frac{1}{6}\sum_{\nu=0}^{5} L_+^{5-\nu} L_3 L_+^{\nu} = \frac{1}{6}\big[6L_+^5 L_3 + 15L_+^5\big],$$

$$\mathrm{Sym}(L_-^5 L_3) = \frac{1}{6}\sum_{\nu=0}^{5} L_-^{\nu} L_3 L_-^{5-\nu} = \frac{1}{6}[6L_3 L_-^5 + 15L_-^5],$$

$$\mathrm{Sym}\big(L_+^5 L_3\big) + \mathrm{Sym}(L_-^5 L_3) = \frac{1}{6}\big[6L_+^5 L_3 + 15L_+^5 + 6L_3 L_-^5 + 15L_-^5\big]. \tag{A.1}$$

The next term of $\mathcal{K}$ offers no problem, we find $\mathrm{Sym}\big(L_3^6\big) = L_3^6$.

The following three terms of $\mathcal{K}$ have equal powers in (L_+, L_-). Therefore they can be expressed as polynomial functions of the commuting operators (L^2, L_3). Consider the term of $\mathcal{K}$ quadratic in (L_+, L_-). These two operators can appear in two orders which for short we denote as

$$(AB) = (+-), (-+). \tag{A.2}$$

Using the commutator and the Casimir invariant L^2, we get the well-known results

$$L_+L_- = L^2 - L_3(L_3 - 1), \qquad L_-L_+ = L^2 - L_3(L_3 + 1). \tag{A.3}$$

Both operators are diagonal with respect to states labelled by (j, m). It proves convenient to pass to the matrix elements by writing

$$\begin{aligned} &a(m) := \langle jm|L_+L_-|jm\rangle = j(j+1) - m(m-1), \\ &a(m+1) = \langle jm|L_-L_+|jm\rangle = j(j+1) - m(m+1), \\ &a(m) = 0 \text{ if } m < -j, m > j. \end{aligned} \tag{A.4}$$

The coefficients $a(m)$ (equation (A.4)) will appear in the following equations of this section. For short, we suppress their dependence on the fixed irrep label j. In the full term of $\mathcal{K}$ quadratic in (L_+, L_-) we must now insert four powers of $L_3 := C$. In Sym there appear 15 monomial terms of the order $\ldots A \ldots B \ldots$. We order them as

$$\begin{array}{lllll} (C^4AB), & (C^3ACB), & (C^3ABC), & (C^2AC^2B), & (C^2ACBC), \\ (C^2ABC^2), & (CAC^3B), & (CAC^2BC), & (CACBC^2), & (CABC^3), \\ (AC^4B), & (AC^3BC), & (AC^2BC^2), & (ACBC^3), & (ABC^4). \end{array} \tag{A.5}$$

When we pass to the diagonal matrix elements of the terms in equation (A.5), the powers of $L_3 = C$ contribute quartic expressions in m whose values depend on the choice of (AB) and on the order (equation (A.5)) of insertion. A straightforward evaluation in terms of $a(m)$ (equation (A.4)), m yields for the full sum of all these 30 terms

$$\begin{aligned} \langle jm|\mathrm{Sym}\big(L_+L_-L_3^4\big)|jm\rangle &= \tfrac{1}{30}a(m)(15m^4 - 20m^3 + 15m^2 - 6m + 1) \\ &\quad + \tfrac{1}{30}a(m+1)(15m^4 + 20m^3 + 15m^2 + 6m + 1). \end{aligned} \tag{A.6}$$

The evaluation of the monomials in $\mathcal{K}$ of power 4 in (L_+, L_-) proceeds in the same fashion. First, we order the powers of (L_+, L_-) in shorthand notation as

$$\begin{aligned} (ABDE) = {} & (++--), (+-+-), (+--+), \\ & (-++-), (-+-+), (--++). \end{aligned} \tag{A.7}$$

For any fixed order $(ABDE)$ chosen from equation (A.7), the two additional powers of $C = L_3$ can be inserted according to the 15 terms

$$\begin{array}{lll} (C^2ABDE), & (CACBDE), & (CABCDE), \\ (CABDCE), & (CABDEC), & (AC^2BDE), \\ (ACBCDE), & (ACBDCE), & (ACBDEC), \\ (ABC^2DE), & (ABCDCE), & (ABCDEC), \\ (ABDC^2E), & (ABDCEC), & (ABDEC^2). \end{array} \tag{A.8}$$

The powers of L_3 contribute quadratic expressions in m, and the full sum of 90 monomial terms, after summing in each one over the 15 terms (equation (A.8)), reduces in terms of $a(m)$ from equation (A.4) to

$$\begin{aligned}\langle jm|\mathrm{Sym}\big(L_+^2L_-^2L_3^2\big)|jm\rangle &= \tfrac{1}{90}a(m-1)a(m)(15m^2-24m+11)\\ &\quad+\tfrac{1}{90}a(m)a(m)(15m^2-12m+3)+\tfrac{1}{90}a(m)a(m+1)(15m^2+1)\\ &\quad+\tfrac{1}{90}a(m+1)a(m)(15m^2+1)+\tfrac{1}{90}a(m+1)a(m+1)(15m^2+12m+3)\\ &\quad+\tfrac{1}{90}a(m+2)a(m+1)(15m^2+24m+11).\end{aligned} \tag{A.9}$$

We keep all six terms in correspondence to equation (A.7).

Finally, we evaluate from $\mathcal{K}$ the sum of monomials of power 6 in (L_+, L_-). The 20 orderings can be abridged as

$$\begin{aligned}(ABDEFG) =&\\ &(+++---),(++-+--),(++--+-),(++---+),\\ &(+-++--),(+-+-+-),(+-+--+),(+--++-),\\ &(+--+-+),(+---++),(-+++--),(-++-+-),\\ &(-++--+),(-+-++-),(-+-+-+),(-+--++),\\ &(--+++-),(--++-+),(--+-++),(---+++).\end{aligned} \tag{A.10}$$

The evaluation yields in the order of equation (A.10)

$$\begin{aligned}\langle jm|\mathrm{Sym}\big(L_+^3L_-^3\big)|jm\rangle = \tfrac{1}{20}&\\ &(a(m-2)\cdot a(m-1)\cdot a(m)+a(m-1)\cdot a(m-1)\cdot a(m)\\ &+a(m-1)\cdot a(m)\cdot a(m)+a(m-1)\cdot a(m)\cdot a(m+1)\\ &+a(m)\cdot a(m-1)\cdot a(m)+a(m)\cdot a(m)\cdot a(m)\\ &+a(m)\cdot a(m)\cdot a(m+1)+a(m)\cdot a(m+1)\cdot a(m)\\ &+a(m)\cdot a(m+1)\cdot a(m+1)+a(m)\cdot a(m+2)\cdot a(m+1)\\ &+a(m+1)\cdot a(m-1)\cdot a(m)+a(m+1)\cdot a(m)\cdot a(m)\\ &+a(m+1)\cdot a(m)\cdot a(m+1)+a(m+1)\cdot a(m+1)\cdot a(m)\\ &+a(m+1)\cdot a(m+1)\cdot a(m+1)+a(m+1)\cdot a(m+2)\cdot a(m+1)\\ &+a(m+2)\cdot a(m+1)\cdot a(m)+a(m+2)\cdot a(m+1)\cdot a(m+1)\\ &+a(m+2)\cdot a(m+2)\cdot a(m+1)+a(m+3)\cdot a(m+2)\cdot a(m+1)).\end{aligned} \tag{A.11}$$

One can easily convert all expectation values back into operators by inserting the commuting operators (L^2, L_3) in equations (A.6), (A.9) and (A.11). It can be recognized that the terms in $\mathcal{K}$ of highest power in (L_+, L_3, L_-) correspond to the commutative invariant (equation (44)). All other terms reflect the non-commutative structure of $\mathcal{K}$.

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